Quadratic Transportation Cost Inequalities For Stochastic Reaction Diffusion Equations Driven by Multiplicative Space-Time White Noise

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Abstract

We established a quadratic transportation cost inequality for solutions of stochastic reaction diffusion equations driven by multiplicative space-time white noise based on a new inequality we proved for the moments (under the uniform norm) of the stochastic convolution with respect to space-time white noise, which is of independent interest. This talk is based on two papaers.

1. Shijie Shang and Tusheng Zhang: Quadratic transportation cost inequality for stochastic reaction diffusion equations driven by multiplicative space-time white noise, preprint 2019.

2. Fengyu Wang and Tusheng Zhang: Talagrand Inequality on Free Path Space and Application to Stochastic Reaction Diffusion Equations, preprint 2019.

Introduction and framework

Let (X, d) be a metric space with a Borel probability measure μ. For a measurable subset $A \subset X$ and $r > 0$, we denote by A_r the r-neighborhood of A, namely $A_r = \{x : d(x, A) < r\}$. We say that μ has normal concentration on (X, d) if there are constants $C, c > 0$ such that for every $r > 0$ and every Borel subset A with $\mu(A) \geq \frac{1}{2}$,

$$1 - \mu(A_r) \leq Ce^{-cr^2}. \tag{1}$$

It is well known that Gaussian measures on $\mathbb{R}^d$ and uniform measures on the spheres $\mathbb{S}^d$ have normal concentration. In the past decades, many people established normal concentration properties for various kinds of interesting measures. We mention the celebrated works of M. Talagrand [T1], [T2] and [T3]. We refer the readers to the monograph [L] for a nice exposition of the concentration of measure phenomenon.

Introduction and framework

It turns out that the concentration of measure phenomenon has close connections with entropy and functional inequalities, e.g. Poincare inequalities, logarithmic Sobolev inequalities and transportation cost inequalities. In particular, transportation cost inequalities imply the normal concentration. An elegant, simple proof of this fact is contained in the book [L]. The importance of the topic of the concentration of measure lies also in its wide applications, e.g. to stochastic finance (see [La]), statistics (see [M]) and the analysis of randomized algorithms (see [DP]).

Introduction and framework

The concentration of measure for stochastic differential equations and stochastic partial differential equations (SPDEs) has been investigated by many people. Let us mention several papers which are relevant to our work. The transportation cost inequalities for stochastic differential equations were obtained by H. Djellout, A. Guillin and L. Wu in [DGW]. The measure concentration for multidimensional diffusion processes with reflecting boundary conditions was considered by S. Pal in [P]. Transportation cost inequalities for solution of stochastic partial differential equations driven by Gaussian noise which is white in time and colored in space were obtained by A. S. Ustunel in [U]. We particularly like to mention the paper [KS] by D. Khoshnevisan and A. Sarantsev, which is the starting point of our work.

Introduction and framework

In [KS], the authors established the quadratic transportation cost inequality under L^2-distance for stochastic reaction diffusion equations driven by multiplicative space-time white noise. However, under the uniform distance they only obtained the quadratic transportation cost inequality for stochastic reaction diffusion equations driven by additive space-time white noise. As is well known, one of the essential differences between SPDEs driven by colored noise and SPDEs driven by space-time white noise is that the solution of the later is not a semimartingale and therefore in particular Ito formula could not be used.

Introduction and framework

Our aim is to prove that under the uniform distance the quadratic transportation cost inequality holds for stochastic reaction diffusion equations driven by multiplicative space-time white noise. Our new contribution is the pth moment inequalities under the uniform norm we obtained for the stochastic convolution with respect to space-time white noise, which is of independent interest. The significance of the inequality is to allow the order p of the moment to be any positive number, not just for sufficiently large ones. These new estimates allow us to establish the quadratic transportation cost inequality under the uniform norm.

Introduction and framework

In the second part of my talk, I will briefly mention the new progress where we obtained the quadratic transportation cost inequality holds for stochastic reaction diffusion equations with random initial values. This is done as an application of a general result on the quadratic transportation cost inequality of Markov processes on free path spaces.

Introduction and framework

Let (X, d, μ) be a metric space with a Borel probability measure μ. The concentration function $\alpha_\mu(r)$ is defined as

$$\alpha_\mu(r) := \sup\left\{1 - \mu(A_r) : A \subset X, \mu(A) \geq \frac{1}{2}\right\}, \quad r > 0.$$

The normal concentration of μ means that $\alpha_\mu(r) \leq Ce^{-cr^2}$ for all $r > 0$ with some positive constants C, c.
Let μ, ν be two Borel probability measures on the metric space (X, d). Consider the Wasserstein distance

$$W_2(\nu, \mu) := \left[\inf \int_X \int_X d(x, y)^2 \pi(dx, dy)\right]^{\frac{1}{2}}$$

Introduction and framework

between μ and ν, where the infimum is taken over all probability measures π on the product space $X \times X$ with marginals μ and ν. Recall that the relative entropy of ν with respect to μ is defined by

$$H(\nu|\mu) := \int_X log(\frac{d\nu}{d\mu})d\nu,$$

if ν is absolutely continuous with respect to μ, and $+\infty$ if not. We say that the measure μ satisfies a quadratic transportation cost inequality if there exists a constant $C > 0$ such that for all probability measures ν,

$$W_2(\nu, \mu) \leq C\sqrt{H(\nu|\mu)}. \qquad (2)$$

The following result is taken from [L].

Proposition 1 If μ satisfies a quadratic transportation cost inequality, then μ has normal concentration.

Remark. The notion of concentration of measure phenomenon depends on the underlying topology of the associated metric space. The stronger the topology, the stronger the concentration.

Now consider the following equation:

$$\begin{cases} du(t,x) = \frac{1}{2}u''(t,x)dt + b(u(t,x))dt + \sigma(u(t,x))W(dt,dx), \\ u(t,0) = u(t,1) = 0, \qquad t > 0, \\ u(0,x) = u_0(x), \quad x \in (0,1), \end{cases} \tag{3}$$

where $u_0 \in C_0(0,1)$, $W(dt, dx)$ is a space-time white noise on some filtrated probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, here $\mathcal{F}_t, t \geq 0$ are the argumented filtration generated by the Brownian sheet $\{W(t,x); (t,x) \in [0,\infty) \times [0,1]\}$.

The coefficients $b(\cdot), \sigma(\cdot) : R \to R$ are deterministic measurable functions. We say that an adapted, continuous random field $\{u(t,x) : (t,x) \in R_+ \times [0,1]\}$ is a solution to the stochastic partial differential equation (SPDE) (3) if $t \geq 0$,

$$\int_0^1 u(t,x)\phi(x)dx = \int_0^1 u_0(x)\phi(x)dx + \frac{1}{2}\int_0^t ds \int_0^1 u(s,x)\phi''(x)dx + \int_0^t ds \int_0^1 b(u(s,x))\phi(x)dx + \int_0^t \int_0^1 \sigma(u(s,x))\phi(x)W(ds,dx), \quad P-a.s. \tag{4}$$

for any $\phi \in C_0^2(0,1)$.

It is well known (see [W]) that u is a solution to SPDE (3) if and only if u satisfies the following integral equation

$$\begin{aligned} u(t,x) =& P_t u_0(x) + \int_0^t \int_0^1 p_{t-s}(x,y) b(u(s,y)) ds dy \\ &+ \int_0^t \int_0^1 p_{t-s}(x,y) \sigma(u(s,y)) W(ds,dy), \end{aligned} \tag{5}$$

where $P_t, t \geq 0$ and $p_t(x,y)$ are the corresponding semigroup and the heat kernel associated with the operator $\frac{1}{2}\Delta$ equipped with the Dirichlet boundary condition on the interval $[0,1]$.

Introduction and framework

Introduce the hypotheses

(H.1) There exists a constant L_b such that for all $x, y \in \mathbb{R}$,

$$|b(x)| \leq L_b(1 + |x|),$$
$$|b(x) - b(y)| \leq L_b|x - y|. \tag{6}$$

(H.2) There exist constants K_σ and L_σ such that for all $x, y \in \mathbb{R}$,

$$|\sigma(x)| \leq K_\sigma,$$
$$|\sigma(x) - \sigma(y)| \leq L_\sigma|x - y|. \tag{7}$$

It is well known (see [W]) that under the hypotheses (H.1) and (H.2), SPDE (3) admits a unique random field solution $u(t, x)$. In fact, the boundedness of the diffusion coefficient $\sigma(\cdot)$ is necessary for proving the transportation cost inequality.

Moment estimates for stochastic convolution

As an important part of the work, we prove some new moment estimates for the stochastic convolution against space-time white noise. Of particular interest are the estimates of the moments of lower order. These bounds play a crucial role.

Proposition 2 Let $\{\sigma(s,y) : (s,y) \in R_+ \times [0,1]\}$ be a random field such that the stochastic integral against space time white noise is well defined. Then for any $T > 0$, $p > 10$, there exists a constant $C_{T,p} > 0$ such that

$$\begin{aligned} &E\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)W(ds,dy)\right|^p\right] \\ \leq & C_{T,p}\int_0^T \sup_{y\in[0,1]} E\left|\sigma(s,y)\right|^p ds. \end{aligned} \tag{8}$$

Moment estimates for stochastic convolution

Sketch of the proof. We employ the factorization method. Choose α such that $\frac{3}{2p} < \alpha < \frac{1}{4} - \frac{1}{p}$. This is possible because $p > 10$. Let

$$(J_\alpha \sigma)(s,y) := \int_0^s \int_0^1 (s-r)^{-\alpha} p_{s-r}(y,z)\sigma(r,z)W(dr,dz), \tag{9}$$

$$(J^{\alpha-1} f)(t,x) := \frac{sin\pi\alpha}{\pi} \int_0^t \int_0^1 (t-s)^{\alpha-1} p_{t-s}(x,y) f(s,y) dsdy. \tag{10}$$

By the stochastic Fubini theorem, for any $(t,x) \in \mathbb{R}_+ \times [0,1]$,

$$\int_0^t \int_0^1 p_{t-s}(x,y)\sigma(s,y)W(ds,dy) = J^{\alpha-1}(J_a \sigma)(t,x). \tag{11}$$

Moment estimates

Therefore

$$\sup_{(t,x)\in[0,T]\times[0,1]} \left| \int_0^t \int_0^1 p_{t-s}(x,y)\sigma(s,y)W(ds,dy) \right|$$
$$= \sup_{(t,x)\in[0,T]\times[0,1]} \left| J^{\alpha-1}(J_\alpha \sigma)(t,x) \right|, \quad P-a.s.. \tag{12}$$

Moment estimates

Using the Höler's inquality, we have

$$E \sup_{(t,x)\in[0,T]\times[0,1]} \left| \int_0^t \int_0^1 p_{t-s}(x,y)\sigma(s,y)W(ds,dy) \right|^p$$

$$\leq \cdots$$

$$\leq \left|\frac{sin\pi\alpha}{\pi}\right|^p E \sup_{(t,x)\in[0,T]\times[0,1]} \left\{ \int_0^t (t-s)^{\alpha-1} \times \left(\int_0^1 p_{t-s}(x,y)^2 \, dy \right)^{\frac{1}{2}\times\frac{2}{p}} \left(\int_0^1 |J_\alpha\sigma(s,y)|^p \, dy \right)^{\frac{1}{2}\times\frac{2}{p}} ds \right\}^p$$

$$\leq \cdots$$

$$\leq \left|\frac{sin\pi\alpha}{\pi}\right|^p C_2 \times \int_0^T \int_0^1 E|J_\alpha\sigma(s,y)|^p \, dyds$$

$$\leq C'_{T,p} \sup_{(s,y)\in[0,T]\times[0,1]} E \left| \int_0^s \int_0^1 (s-r)^{-\alpha} p_{s-r}(y,z)\sigma(r,z)W(dr,dz) \right|^p,$$

Moment estimates

Applying the BDG inequality, we have

$$\begin{aligned}
&\left\| \int_0^s \int_0^1 (s-r)^{-\alpha} p_{s-r}(y,z)\sigma(r,z)W(dr,dz) \right\|_{L^p(\Omega)}^2 \\
\leq& 4p \int_0^s \int_0^1 (s-r)^{-2\alpha} p_{s-r}(y,z)^2 \|\sigma(r,z)\|_{L^p(\Omega)}^2 \, drdz \\
\leq& 4p \int_0^s (s-r)^{-2\alpha} \left(\int_0^1 p_{s-r}(y,z)^2 \, dz \right) \sup_{z\in[0,1]} \|\sigma(r,z)\|_{L^p(\Omega)}^2 \, dr \\
\leq& 4C_2 p \int_0^s (s-r)^{-2\alpha-\frac{1}{2}} \sup_{z\in[0,1]} \|\sigma(r,z)\|_{L^p(\Omega)}^2 \, dr \\
\leq& 4C_2 p \left(\int_0^s (s-r)^{(-2\alpha-\frac{1}{2})\times\frac{p}{p-2}} \, dr \right)^{\frac{p-2}{p}} \times \left(\int_0^s \sup_{z\in[0,1]} \|\sigma(r,z)\|_{L^p(\Omega)}^p \, dr \right)
\end{aligned} \tag{14}$$

Moment estimates

Therefore

$$\sup_{(s,y)\in[0,T]\times[0,1]} E\left|\int_0^s\int_0^1 (s-r)^{-\alpha}p_{s-r}(y,z)\sigma(r,z)W(dr,dz)\right|^p$$

$$\leq C''_{T,p}\times\int_0^T \sup_{z\in[0,1]} E\left|\sigma(r,z)\right|^p\, dr, \tag{15}$$

where the condition $\alpha<\frac{1}{4}-\frac{1}{p}$ was used.

Moment estimates

Combining (13) with (15), we obtain

$$
\begin{aligned}
& E \sup_{(t,x)\in[0,T]\times[0,1]} \left| \int_0^t \int_0^1 p_{t-s}(x,y)\sigma(s,y)W(ds,dy) \right|^p \\
\leq & C_{T,p} \int_0^T \sup_{z\in[0,1]} E \left|\sigma(r,z)\right|^p \, dr, \qquad (16)
\end{aligned}
$$

where

$$
C_{T,p} = \min_{\frac{3}{2p}<\alpha<\frac{1}{4}-\frac{1}{p}} C'_{T,p,\alpha} \times C''_{T,p,\alpha}. \qquad (17)
$$

Lemma 3 Let $\sigma(s,y)$ be as in Proposition 2, then for any $T>0$, $p>10$, $\lambda>0$, there exists a constant $C_{T,p}>0$ such that

$$
\begin{aligned}
&P\left(\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)W(ds,dy)\right|>\lambda\right)\\
\leq &P\left(\int_0^T \sup_{y\in[0,1]}|\sigma(s,y)|^p\,ds>\lambda^p\right)\\
&+\frac{C_{T,p}}{\lambda^p}E\min\left\{\lambda^p,\int_0^T\sup_{y\in[0,1]}|\sigma(s,y)|^p\,ds\right\}. \qquad (18)
\end{aligned}
$$

Here the constant $C_{T,p}$ is the same as the constant $C_{T,p}$ in (8).

Sketch of the proof. For any $\lambda > 0$, define

$$\Omega_\lambda := \left\{ \omega \in \Omega : \int_0^T \sup_{y \in [0,1]} |\sigma(s,y)|^p \, ds \le \lambda^p \right\}. \qquad (19)$$

By Chebyshev's inequality, we have

Moment estimates

$$
\begin{aligned}
&P\left(\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)W(ds,dy)\right|>\lambda\right)\\
\leq&P(\Omega\backslash\Omega_\lambda)\\
&+P\left(\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)W(ds,dy)\right|I_{\Omega_\lambda}>\lambda\right)\\
\leq&P(\Omega\backslash\Omega_\lambda)\\
&+\frac{1}{\lambda^p}E\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|I_{\Omega_\lambda}\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)W(ds,dy)\right|^p\right].
\end{aligned}
\tag{20}
$$

Now, we introduce the random field

$$\widetilde{\sigma}(s,y) := \sigma(s,y) I_{\left\{\omega\in\Omega:\ \int_0^s \sup_{y\in[0,1]} |\sigma(r,y)|^p dr \le \lambda^p\right\}}. \qquad (21)$$

Since for any $\omega \in \Omega_\lambda$,

$$\int_0^t \int_0^1 |\sigma(s,y) - \widetilde{\sigma}(s,y)|^2 \, dsdy = 0, \quad \forall\, t \in [0,T], \qquad (22)$$

by the local property of the stochastic integral,

$$\begin{aligned} & I_{\Omega_\lambda} \int_0^t \int_0^1 p_{t-s}(x,y)\sigma(s,y) W(ds,dy) \\ = & I_{\Omega_\lambda} \int_0^t \int_0^1 p_{t-s}(x,y)\widetilde{\sigma}(s,y) W(ds,dy), \quad P-a.s.. \qquad (23) \end{aligned}$$

Moment estimates

Hence using the bound (8), we get

$$
\begin{aligned}
& E\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|I_{\Omega_\lambda}\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)W(ds,dy)\right|^p\right] \\
\leq & E\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\widetilde{\sigma}(s,y)W(ds,dy)\right|^p\right] \\
\leq & C_{T,p}E\int_0^T\sup_{y\in[0,1]}|\widetilde{\sigma}(s,y)|^p\ ds \\
\leq & C_{T,p}E\min\left\{\lambda^p,\int_0^T\sup_{y\in[0,1]}|\sigma(s,y)|^p\ ds\right\}.
\end{aligned} \tag{24}
$$

Combining (20) with (24), we obtain (18).

Moment estimates

Proposition 4 Let $\{\sigma(s,y) : (s,y) \in \mathbb{R}_+ \times [0,1]\}$ be a random field such that the stochastic integral against space time white noise is well defined. Then the following two estimates hold:

(i) for any $T > 0$, $0 < p \leq 10$, $q > 10$, there exists a constant $C_{T,p,q}$ such that

$$E\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)W(ds,dy)\right|^p\right]$$
$$\leq C_{T,p,q}E\left[\int_0^T \sup_{y\in[0,1]}|\sigma(s,y)|^q\, ds\right]^{\frac{p}{q}}. \tag{25}$$

Moment estimates

(ii) For any $T > 0$, $0 < p \leq 10$, $\epsilon > 0$, there exists a constant $C_{T,p,\epsilon}$ such that

$$E\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)W(ds,dy)\right|^p\right]$$
$$\leq \epsilon E\left[\sup_{(s,y)\in[0,T]\times[0,1]}|\sigma(s,y)|^p\right] + C_{T,p,\epsilon}E\int_0^T \sup_{y\in[0,1]}|\sigma(s,y)|^p\, ds. \tag{26}$$

Moment estimates

Sketch of the proof. The estimate (25) can be derived as follows:

$$
\begin{aligned}
& E\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)W(ds,dy)\right|^p\right] \\
=& \int_0^\infty p\lambda^{p-1}P\left(\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)W(ds,dy)\right|>\lambda\right) \\
\le& \int_0^\infty p\lambda^{p-1}P\left(\int_0^T\sup_{y\in[0,1]}|\sigma(s,y)|^q\,ds>\lambda^q\right)d\lambda \\
& + C_{T,p}\int_0^\infty p\lambda^{p-1-q}E\min\left\{\lambda^q,\int_0^T\sup_{y\in[0,1]}|\sigma(s,y)|^q\,ds\right\}d\lambda \\
=& C_{T,p,q}E\left[\int_0^T\sup_{y\in[0,1]}|\sigma(s,y)|^q\,ds\right]^{\frac{p}{q}}. \qquad (27)
\end{aligned}
$$

Moment estimates

The assertion (ii) can be obtained using Young inequality as follows: From (25) it follows that for any $q > 10$,

$$
\begin{aligned}
&E\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)W(ds,dy)\right|^p\right] \\
\leq &C_{T,p,q}E\left[\int_0^T \sup_{y\in[0,1]}|\sigma(s,y)|^q\,ds\right]^{\frac{p}{q}} \\
\leq &C_{T,p,q}E\left[\sup_{(s,y)\in[0,T]\times[0,1]}|\sigma(s,y)|^{q-p}\times\int_0^T\sup_{y\in[0,1]}|\sigma(s,y)|^p\,ds\right]^{\frac{p}{q}} \\
\leq &\epsilon E\left[\sup_{(s,y)\in[0,T]\times[0,1]}|\sigma(s,y)|^p\right]+C_{T,p,q,\epsilon}E\int_0^T\sup_{y\in[0,1]}|\sigma(s,y)|^p\,ds.
\end{aligned}
\tag{28}
$$

Quadratic transportation cost inequality

Let μ be the law of the random field solution $u(\cdot, \cdot)$ of SPDE (3), viewed as a probability measure on $C([0, T] \times [0, 1])$. Our main result says that μ satisfies a quadratic transportation cost inequality under the uniform norm.

Quadratic transportation cost inequality

First we recall a lemma proved in [KS] describing the probability measures ν that are absolutely continuous with respect to μ.

Let $\nu \ll \mu$ on $C([0,T] \times [0,1])$. Define a new probability measure Q on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \le t \le T}, P)$ by

$$dQ := \frac{d\nu}{d\mu}(u) dP. \tag{29}$$

Denote the Radon-Nikodym derivative restricted on $\mathcal{F}_t$ by

$$M_t := \left. \frac{dQ}{dP} \right|_{\mathcal{F}_t}, \quad t \in [0,T].$$

Then $M_t, t \in [0,T]$ forms a P-martingale. The following result was proved in [KS].

Lemma 5. There exists an adapted random field $h = \{h(s,x), (s,x) \in [0,T] \times [0,1]\}$ such that $Q - a.s.$ for all $t \in [0,T]$,

$$\int_0^t \int_0^1 h^2(s,x) ds dx < \infty$$

and $\widetilde{W} : [0,T] \times [0,1] \to \mathbb{R}$ defined by

$$\widetilde{W}(t,x) := W(t,x) - \int_0^t \int_0^x h(s,y) ds dy, \tag{30}$$

is a Brownian sheet under the measure Q.

Quadratic transportation cost inequality

Moreover,

$$M_t = \exp\left(\int_0^t \int_0^1 h(s,x)W(ds,dx) - \frac{1}{2}\int_0^t \int_0^1 h^2(s,x)dsdx\right), \quad Q - a. \tag{31}$$

and

$$H(\nu|\mu) = \frac{1}{2}E^Q\left[\int_0^T \int_0^1 h^2(s,x)dsdx\right], \tag{32}$$

where E^Q stands for the expectation under the measure Q.

Quadratic transportation cost inequality

The following theorem is the main result.
Theorem 6 Suppose the hypotheses (H.1) and (H.2) hold. Then the law μ of the solution $u(\cdot,\cdot)$ of SPDE (3) satisfies the quadratic transportation cost inequality on the space $C([0,T]\times[0,1])$. Consequently μ has normal concentration.

Quadratic transportation cost inequality

Take $\nu \ll \mu$ on $C([0,T]\times[0,1])$. Define the corresponding measure Q by (29). Let $h(t,x)$ be the corresponding random field appeared in Lemma 5. Then the solution $u(t,x)$ of equation (3) satisfies the following SPDE under the measure Q,

$$\begin{aligned} u(t,x) =& P_t u_0(x) + \int_0^t \int_0^1 p_{t-s}(x,y) b(u(s,y))\, dsdy \\ &+ \int_0^t \int_0^1 p_{t-s}(x,y)\sigma(u(s,y))\widetilde{W}(ds,dy) \\ &+ \int_0^t \int_0^1 p_{t-s}(x,y)\sigma(u(s,y))h(s,y)\, dsdy. \qquad (33) \end{aligned}$$

Consider the solution of the following SPDE:

$$v(t,x) = P_t u_0(x) + \int_0^t \int_0^1 p_{t-s}(x,y) b(v(s,y))\, dsdy + \int_0^t \int_0^1 p_{t-s}(x,y)\sigma(v(s,y)) \widetilde{W}(ds,dy). \qquad (34)$$

By Lemma 5 it follows that under the measure Q, the law of (v,u) forms a coupling of (μ,ν).

Quadratic transportation cost inequality

Therefore by the definition of the Wasserstein distance,

$$W_2(\nu,\mu)^2 \le E^Q\left[\sup_{(t,x)\in[0,T]\times[0,1]} |u(t,x)-v(t,x)|^2\right].$$

In view of (32), to prove the quadratic transportation cost inequality

$$W_2(\nu,\mu) \le \sqrt{2CH(\nu|\mu)}, \tag{35}$$

it is sufficient to show that

$$E^Q\left[\sup_{(t,x)\in[0,T]\times[0,1]} |v(t,x)-u(t,x)|^2\right] \le CE^Q\left[\int_0^T\int_0^1 h^2(s,y)\,dsdy\right]. \tag{36}$$

for some independent constant C.

Quadratic transportation cost inequality

For simplicity, in the sequel we still denote E^Q by the symbol E. It follows that

$$E\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|v(t,x)-u(t,x)\right|^2\right]\leq 3(I+II+III), \tag{37}$$

where

$$I := E\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\big[b(v(s,y))-b(u(s,y))\big]\,dsdy\right|^2\right]$$

Quadratic transportation cost inequality

$$
\begin{aligned}
II := & E\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\big[\sigma(v(s,y))\right.\right.\\
& \left.\left.-\sigma(u(s,y))\big]\tilde{W}(ds,dy)\right|^2\right],\\
III := & E\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(u(s,y))h(s,y)\,dsdy\right|^2\right].
\end{aligned}
$$

Quadratic transportation cost inequality

By Holder's inequality, the term I can be estimated as follows:

$$\begin{aligned}
I \leq & L_b^2 E\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)|v(s,y)-u(s,y)|\,dsdy\right|^2\right]\\
\leq & L_b^2 E\Bigg\{\sup_{(t,x)\in[0,T]\times[0,1]}\Bigg[\left(\int_0^t\int_0^1 p_{t-s}(x,y)^2\,dsdy\right)\\
& \qquad\qquad \times\left(\int_0^t\int_0^1 |v(s,y)-u(s,y)|^2\,dsdy\right)\Bigg]\Bigg\}\\
\leq & \sqrt{\frac{2T}{\pi}}L_b^2 E\int_0^T\int_0^1 |v(s,y)-u(s,y)|^2\,dsdy\\
\leq & \sqrt{\frac{2T}{\pi}}L_b^2\int_0^T E\left[\sup_{(r,y)\in[0,s]\times[0,1]}|v(r,y)-u(r,y)|^2\right]ds. \qquad (38)
\end{aligned}$$

Quadratic transportation cost inequality

For the term II, applying the estimate (26) we obtain that for any $\epsilon > 0$,

$$
\begin{aligned}
II \leq & \epsilon E\left[\sup_{(t,x)\in[0,T]\times[0,1]} |\sigma(v(t,x)) - \sigma(u(t,x))|^2\right] \\
& + C_{T,2,\epsilon} E \int_0^T \sup_{y\in[0,1]} |\sigma(v(s,y)) - \sigma(u(s,y))|^2 \, ds \\
\leq & \epsilon L_\sigma^2 E\left[\sup_{(t,x)\in[0,T]\times[0,1]} |v(t,x) - u(t,x)|^2\right] \\
& + C_{T,2,\epsilon} L_\sigma^2 \int_0^T E\left[\sup_{(r,y)\in[0,s]\times[0,1]} |v(r,y) - u(r,y)|^2\right] ds.
\end{aligned}
\tag{39}
$$

Quadratic transportation cost inequality

The term III can be bounded as follows:

$$III \leq K_\sigma^2 E\left\{ \sup_{(t,x)\in[0,T]\times[0,1]} \left[\left(\int_0^t \int_0^1 p_{t-s}(x,y)^2 \, dsdy \right) \times \left(\int_0^t \int_0^1 h^2(s,y) \, dsdy \right) \right] \right\}$$

$$\leq \sqrt{\frac{2T}{\pi}} K_\sigma^2 E\left[\int_0^T \int_0^1 h^2(s,y) \, dsdy \right]. \tag{40}$$

Set

$$Y(t) := E\left[\sup_{(s,x)\in[0,t]\times[0,1]} |v(s,x) - u(s,x)|^2 \right]. \tag{41}$$

Quadratic transportation cost inequality

Putting (37)-(40) together, we obtain

$$
\begin{aligned}
Y(T) \leq & 3\sqrt{\frac{2T}{\pi}} L_b^2 \int_0^T Y(s)\, ds + 3\epsilon L_\sigma^2 Y(T) + 3C_{T,2,\epsilon} L_\sigma^2 \int_0^T Y(s)\, ds \\
& + 3\sqrt{\frac{2T}{\pi}} K_\sigma^2 E\left[\int_0^T \int_0^1 h^2(s,y)\, dsdy\right]. \qquad (42)
\end{aligned}
$$

Quadratic transportation cost inequality

Taking any $\epsilon < \frac{1}{3L_\sigma^2}$, we deduce from (42) that

$$Y(T) \leq \frac{3L_b^2}{1-3\epsilon L_\sigma^2}\sqrt{\frac{2T}{\pi}}\int_0^T Y(s)\,ds + \frac{3C_{T,2,\epsilon}L_\sigma^2}{1-3\epsilon L_\sigma^2}\int_0^T Y(s)\,ds + \frac{3K_\sigma^2}{1-3\epsilon L_\sigma^2}\sqrt{\frac{2T}{\pi}}E\left[\int_0^T\int_0^1 h^2(s,y)\,dsdy\right]. \qquad (43)$$

Clearly, (43) still holds if we replace T with any $t \in [0, T]$. Applying Gronwall's inequality, we obtain

Quadratic transportation cost inequality

$$
\begin{aligned}
Y(T) \leq & K_\sigma^2 \inf_{0<\epsilon<\frac{1}{3L_\sigma^2}} \left\{ \frac{3}{1-3\epsilon L_\sigma^2} \sqrt{\frac{2T}{\pi}} \exp\left(\frac{3L_b^2 T}{1-3\epsilon L_\sigma^2} \sqrt{\frac{2T}{\pi}} \right.\right. \\
& \left.\left. + \frac{3C_{T,2,\epsilon} L_\sigma^2 T}{1-3\epsilon L_\sigma^2} \right) \right\} \\
& \times E\left[\int_0^T \int_0^1 h^2(s,y)\, ds dy \right]. \qquad (44)
\end{aligned}
$$

This proves (36), hence completes the proof of Theorem 6. ■

Two supplementary results

The following local property of the Walsh stochastic integral against space-time white noise is similar to that of the Ito integral.
Lemma 7. Let $\{\sigma(t,x) : (t,x) \in [0, T] \times [0,1]\}$ be a random field such that the stochastic integral against space time white noise is well defined. Let $\Omega_0 \subset \Omega$ be a measurable subset such that for a.s. $\omega \in \Omega_0$,

$$\int_0^T \int_0^1 |\sigma(t,x)|^2 \, dtdy = 0. \tag{45}$$

Then for a.s. $\omega \in \Omega_0$,

$$\int_0^T \int_0^1 \sigma(t,x) W(dt,dx) = 0. \tag{46}$$

Two supplementary results

Lemma 8 Let $X \geq 0$ be a random variable, then for any $0 < p < q$,

$$EX^p = \int_0^\infty px^{p-1}P(X > x)\,dx, \tag{47}$$

$$\int_0^\infty \frac{E\min\{x^q, X\}}{x^q}px^{p-1}\,dx = \frac{q}{q-p}E\left[X^{\frac{p}{q}}\right]. \tag{48}$$

Part II: Quadratic transportation inequality for Markov processes

Let (E, ρ) be a Polish space, and let $(P_t)_{t \geq 0}$ be the semigroup of a continuous Markov process on E. For any $T > 0$ and $\mu \in \mathcal{P}(E)$, the class of probability measures on the space E, let P^μ denote the distribution of the Markov process up to time T with initial distribution μ; i.e. letting $P_t(x, \cdot)$ be the associated Markov transition kernel, P^μ is the unique probability measure on the free path space

$$E_T := C([0, T]; E) \text{ equipped with } \rho_T(\xi, \eta) := \sup_{t \in [0,T]} \rho(\xi_t, \eta_t),$$

Part II: Quadratic transportation inequality for Markov processes

such that for any $0 = t_0 < t_1 \cdots < t_n = T$ and $\{A_i\}_{0 \le i \le n} \subset \mathcal{B}(E)$,

$$P^\mu(X_{t_i} \in A_i, 0 \le i \le n) = \int_{A_0} \mu(\mathrm{d}x_0) \int_{A_1} P_{t_1 - t_0}(x_0, \mathrm{d}x_1) \cdots \int_{A_n} P_{t_n - t_{n-1}}(x_{n-1}, \mathrm{d}x_n), \tag{49}$$

where $X_t, t \ge 0$ denotes the canonical coordinate process on the path space E_T.

Part II: Quadratic transportation inequality for Markov processes

When $\mu = \delta_x$, the Dirac measure at $x \in E$, we simply denote $P^\mu = P^x$. Then

$$P^\mu = \int_E P^x \mu(\mathrm{d}x), \quad \mu \in \mathcal{P}(E). \tag{50}$$

Let $\mathbb{W}_2$ and $\mathbb{W}_{2,T}$ be the Wasserstein distances induced by ρ on $\mathcal{P}(E)$ and ρ_T on $\mathcal{P}(E_T)$ respectively. We aim to establish the TCI for P^μ by using those for $\{P^x : x \in E\}$ and μ.

Part II: Quadratic transportation inequality for Markov processes

Theorem 9. Assume that for some constants $c_1, c_2 \in (0, \infty)$ one has

$$\mathbb{W}_{2,T}(Q, P^x)^2 \leq c_1 H(Q|P^x), \quad x \in E, Q \in \mathcal{P}(E_T), \tag{51}$$

$$\mathbb{W}_{2,T}(P^x, P^y)^2 \leq c_2 \rho(x,y)^2, \quad x, y \in E. \tag{52}$$

If $\mu \in \mathcal{P}(E)$ satisfies

$$\mathbb{W}_2(\nu, \mu)^2 \leq c_0 H(\nu|\mu), \quad \nu \in \mathcal{P}(E) \tag{53}$$

for some constant $c_0 \in (0, \infty)$, then

$$\mathbb{W}_{2,T}(Q, P^\mu)^2 \leq CH(Q|P^\mu), \quad Q \in \mathcal{P}(E_T) \tag{54}$$

holds for $C = \left(\sqrt{c_1} + \sqrt{c_0 c_2}\right)^2$. On the other hand, (54) implies (53) for $c_0 = C$.

Part II: QTI for SPDEs with random initial values

Let $C_0([0,1]) = \{u \in C([0,1]) : u(0) = u(1) = 0\}$. Consider again the following SPDE on $C_0([0,1])$:

$$\begin{cases} \mathrm{d}u_t(x) = \frac{1}{2}u_t''(x)dt + b(u_t(x))dt + \sigma(u_t(x))W(\mathrm{d}t, \mathrm{d}x), & x \in (0,1), \\ u_t \in C_0([0,1]), \qquad t \geq 0, \end{cases} \tag{55}$$

Part II: QTI for SPDEs with random initial values

Set

$$E := C_0([0,1]), \quad E_T := C([0,T];E) = C([0,T];C_0([0,1])),$$

and let P^μ be the distribution of the solution $(u_t)_{t\in[0,T]}$ with initial distribution $\mu \in \mathcal{P}(E)$. As an application of Theorem 9 we obtain the following

Theorem 10. Let $\mu \in \mathcal{P}(E)$. Then

$$W_2(Q, P^\mu) \le CH(Q|P^\mu), \quad Q \in \mathcal{P}(E_T) \tag{56}$$

holds for some constant $C > 0$ if and only if

$$W_2(\nu, \mu) \le cH(\nu|\mu), \quad \nu \in \mathcal{P}(E) \tag{57}$$

holds for some constant $c > 0$.

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